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STRESSES IN ECCENTRIC STEPWISE DISCONTINUOUS REINFORCING RINGS WITH TRANSITION SECTION

by

Arnold Allentuch and Joseph Kempner

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POLYTECHNIC INSTITUTE OF BROOKLYN

DEPARTMENT
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AEROSPACE ENGINEERING
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ABSTRACT

The stress distribution in a ring of non-uniform cross section under the action of a uniform radial line load is obtained. The ring is fabricated in three segments; one segment whose cross sectional area varies according to a power function and connects two uniform segments. Several sets of parameters are chosen for numerical calculations. Within these sets only the length of the transition section changes. Thus, an appraisal of the importance of the transition section in reducing the maximum stress is made. The stress distribution for each length of the transition section chosen is plotted.

The maximum bending stresses are reduced, because of the transition section, by as much as 63%. The corresponding length of the transition section is approximately 60°.

SECTION 1. LIST OF SYMBOLS

a	Outer (inner) uniform radius, Fig. 1
a_h	Centroidal radius of upper uniform section, Fig. 1
a_f	Centroidal radius of transition section, Fig. 1
a_H	Centroidal radius of lower uniform section, Fig. 1
a_o	Non-dimensional load parameter, Eq. (6)
A_h, A_H	Cross-sectional areas of uniform sections
A_f	Cross-sectional area of transition section
$C_1 \ldots C_4$	Dimensionless constants, Eq. (12)
C_{1_f}	Dimensionless constant, Eq. (12)
C_{3_f}	Dimensionless constant, Eq. (12)
E	Young's modulus
h, H	Height of two uniform sections of ring, Fig. 1
h_f	Height of transition section, Fig. 1
$I_1 \ldots I_5$	Integrals, Eq. (37) to (41)
I_f	Moment of inertia of transition cross section

I_h, I_H	Constant moments of inertia of uniform sections
m_h	Non-dimensional bending moment in lighter uniform section, Eq. (1), Fig. 2
m_f	Non-dimensional bending moment in transition section, Eq. (2), Fig. 3
m_H	Non-dimensional bending moment in heavier uniform section, Eq. (3), Fig. 4
m_o	Non-dimensional bending moment at $\varphi = 0$, Eq. (5), Fig. 2
P	Radial line load distribution on ring
p	Pressure external to cylinder
t	Non-dimensional hoop force, Eq. (9), Fig. 2
t_o	Non-dimensional hoop force at $\varphi = 0$, Eq. (5), Fig. 2
u	Non-dimensional strain energy, Eq. (10)
α	Ratio of centroidal to uniform outer (inner) radius for lighter uniform section, Eq. (7)
γ	Ratio of centroidal to uniform outer (inner) radius for transition section, Eq. (7)
β	Ratio of centroidal to uniform outer (inner) radius for heavier uniform section, Eq. (7)

φ_f	Angle locating beginning of transition section, Fig. 1
φ_o	Angle locating end of transition section, Fig. 1
φ	Angular coordinate measured from crown, Fig. 2
δ	Non-dimensional constant, Eq. (28)
Δ	Non-dimensional constant, Eq. (31)
$(\sigma_\varphi)_b$	Circumferential bending stress
σ_φ	Circumferential membrane stress
ν	Poisson's ratio

2. INTRODUCTION

The general interaction problem of finding the ring and shell stresses in hydrostatically loaded shells, reinforced by periodically spaced frames, has been analyzed in detail in a sequence of PIBAL reports. The initial problems were concerned with uniform rings. In Refs. [1]* and [2] the first solutions of the problem of non-uniform rings mounted on pressurized cylinders was considered. In both of these reports the cross sectional area of the ring was assumed to vary smoothly.

The problem of a hydrostatically loaded shell reinforced by non-uniform stepwise discontinuous rings was investigated in Refs. [3], [4], and [5]. In order better to distinguish between these three solutions, they are henceforth referred to as problems 1, 2, and 3, respectively. Problems 1, 2 and 3 differ only in the geometric configuration of the reinforcing rings.

The rings in problem 1 consisted of two uniform sections having the same mean radius and subtending equal sector angles. Thus the effect of eccentricity of the median line of the ring was not considered.

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*Numbers in square brackets refer to the Bibliography.

In problem 2 the rings were again fabricated of two uniform sections subtending equal sector angles but having <u>different</u> mean radii. The effect of eccentricity was, therefore, included and was found to be significant in the evaluation of the ring bending stress. Several approximations to p roblem 2 were included and evaluated [4].

Finally, the description of the rings in p roblem 3 contained one additional p arameter. These rings conisted of two uniform sections having different mean radii, but the two sections no longer subtended equal sector angles. Thus, the location of the discontinuity, i.e., the junction of the two uniform sections, was a parameter.

In both problems 2 and 3 the maximum bending stress in the ring at the discontinuity was roughly one third of the corresponding membrane stress. In order to investigate the reduction of the bending stress in the ring due to a smoothly varying transition section between the two uniform sections, the interaction problems 2 and 3 have been extended. Let us refer to this as problem 4.

As mentioned previously, several approximations to problem 2 were presented in Ref. [4]. It was concluded that, within the limits of the parameter ranges chosen, these approximations were also valid in problem 3 where the two sections subtended different central angles.

Let us briefly analyze the basis for such a conclusion. In problems 1, 2, and 3 the interaction load, considered as the unknown of the problem, was expressed as a Fourier series whose coefficients were to be determined. In problem 1 the eccentric effect of different mean radii was absent so that the ring bending stress was quite low, i.e., a second order effect. In problem 2, where eccentricity was present, the bending stress was a first order effect, and, in fact, was found to depend most heavily on the constant term of the radial component of the interaction load Fourier series. With this in mind the problem was solved by taking only this constant term into consideration. In effect, the interaction load was assumed to be constant and radial. In Ref. [4] this approximation to problem 2, within the limit of the parameter ranges chosen, was found to be excellent. Further investigation indicated that the approximation described could be further simplified. This was accomplished by solving for the uniform interaction load between an infinitely long hydrostatically loaded cylinder and a single uniform (based on a numerical average of the two cross sectional areas) reinforcing ring. This proved to be a reasonable approximation, within the limits of the chosen parameters, to the constant term of the interaction load for problem 4. One therefore concludes that problem 4 can be reduced to finding the stresses in a uniformly loaded ring composed of two uniform sections and connected by a smoothly varying transition

section. The load (radial) on this ring is obtained from the solution of the interaction problem of a single ring reinforcing an infinitely long hydrostatically loaded shell. In particular, the load may be obtained from Eq. (111), page 62 of Ref. [4].

In the numerical computations that follow, the length of the transition section is the only parameter allowed to change, so that the effect of the transition section can be readily ascertained.

3. THE BENDING MOMENT IN THE RING

The elements of the ring shown in Figs. 2, 3 and 4 are in static equilibrium. From a consideration of moment equilibrium, the results are

$$m_h = m_o - t_o\alpha(1 - \cos\varphi) - \alpha a_o(1 - \cos\varphi) \qquad 0 \leq \varphi \leq \varphi_f \tag{1}$$

$$m_f = m_o - t_o(\alpha - \gamma\cos\varphi) - \gamma\, a_o(1 - \cos\varphi) \qquad \varphi_f \leq \varphi \leq \varphi_o \tag{2}$$

$$m_H = m_o - t_o(\alpha - \beta\cos\varphi) - \beta\, a_o(1 - \cos\varphi) \qquad \varphi_o \leq \varphi \leq \pi \tag{3}$$

where m_h, m_f, and m_H, are bending moments non-dimensionalized with respect to Young's Modulus E and to the constant radius a (see Figs. 1 to 4) as follows:

$$\begin{aligned} m_h &= M_h/Ea^3 \\ m_f &= M_f/Ea^3 \\ m_H &= M_H/Ea^3 \end{aligned} \tag{4}$$

The quantities m_o and t_o are the non-dimensional moment and force, respectively, at $\varphi = 0$.

$$m_o = M_o/Ea^3 \qquad\qquad (5)$$
$$t_o = T_o/Ea^2$$

a_o is a non-dimensionalized measure of the radial line load on the ring P

$$a_o = \frac{P}{Ea} \qquad\qquad (6)$$

Finally, the following non-dimensional quantities in Eqs. (1) to (3) are defined in the following manner (see Fig. 1):

$$\alpha = a_h/a$$
$$\beta = a_H/a \qquad\qquad (7)$$
$$\gamma = a_f/a$$

where a_h, a_f, and a_H are the centroidal radii of the three sections of the ring. While α and β are constants, γ is a function of the angular coordinate φ.

The equations of force equilibrium give the hoop force as a function of the external load

$$t = t_o \cos\varphi - a_o(1 - \cos\varphi) \tag{8}$$

where t is the non-dimensionalized hoop force (see Fig. 1)

$$t = T/Ea^2 \tag{9}$$

The quantities t_o and m_o are unknown. It is convenient at this time to determine these unknowns by Castigliano's theorem. The requirement is that $\frac{\partial u}{\partial t_o} = 0$ and $\frac{\partial u}{\partial m_o} = 0$, where u is a non-dimensionalized strain energy given by

$$u = \frac{U}{Ea^3}\,. \tag{10}$$

If a_m refers to the radius of the median curve of the ring, I the moment of inertia of a cross section, and A the area of a cross section, the strain energy of the ring is

$$u = \int_o^\pi \frac{a_m}{a}\,\frac{a^4}{I}\left(\frac{M}{Ea^3}\right)^2 d\varphi + \int_o^\pi \frac{1}{a_m/a}\,\frac{a^2}{A}\left(\frac{t}{Ea^2}\,\frac{a_m}{a} - \frac{M}{Ea^3}\right)^2 d\varphi \tag{11}$$

At this point let us define the following non-dimensional parameters:

$$C_1 = a^4/I_h \quad C_{1_f} = a^4/I_f \quad C_2 = a^4/I_H$$
$$C_3 = a^2/A_h \quad C_{3_f} = a^2/A_f \quad C_4 = a^2/A_H \tag{12}$$

where the subscripts h, f, and H refer to those sections of the ring having mean radii a_h, a_f, and a_H (see Fig. 1), respectively.

Equation (11) may be written in a manner suitable to this particular problem. Incorporating the definitions (7) and (12) (see Fig. 1 for definition of the position angles φ_f and φ_o), the strain energy may be written as follows:

$$u = \alpha C_1 \int_o^{\varphi_f} m_h{}^2\, d\varphi + \int_{\varphi_f}^{\varphi_o} \gamma C_{1_f}\, m_f{}^2\, d\varphi + \beta C_2 \int_{\varphi_o}^{\pi} m_H{}^2\, d\varphi$$
$$+ \frac{C_3}{\alpha} \int_o^{\varphi_f} (t\alpha - m_h)^2\, d\varphi + \int_{\varphi_f}^{\varphi_o} \frac{C_{3_f}}{\gamma} (t\alpha - m_f)^2\, d\varphi + \frac{C_4}{\beta} \int_{\varphi_o}^{\pi} (t\alpha - m_H)^2\, d\varphi \tag{13}$$

The quantities $(t\alpha - m_h)$, $(t\alpha - m_f)$, and $(t\alpha - m_H)$ can be found from Eqs. (1), (2), (3), and (8) as

$$t\alpha - m_h = t\alpha - m_f = t\alpha - m_H = t_o\alpha - m_o \tag{14}$$

From Eqs. (1), (2), (3), and (14) the following expressions can be readily ascertained:

$$\frac{\partial m_h}{\partial t_o} = -\alpha(1 - \cos\varphi) \tag{15}$$

$$\frac{\partial m_f}{\partial t_o} = -\alpha + \gamma\cos\varphi \tag{16}$$

$$\frac{\partial m_H}{\partial t_o} = - \alpha + \beta \cos \varphi \tag{17}$$

$$\frac{\partial m_h}{\partial m_o} = 1 \tag{18}$$

$$\frac{\partial m_f}{\partial m_o} = 1 \tag{19}$$

$$\frac{\partial m_H}{\partial m_o} = 1 \tag{20}$$

$$\frac{\partial(t\alpha - m_h)}{\partial t_o} = \frac{\partial(t\alpha - m_f)}{\partial t_o} = \frac{\partial(t\alpha - m_H)}{\partial t_o} = \alpha \tag{21}$$

$$\frac{\partial(t\alpha - m_H)}{\partial m_o} = \frac{\partial(t\alpha - m_f)}{\partial m_o} = \frac{\partial(t\alpha - m_H)}{\partial m_o} = - 1 \tag{22}$$

Taking Eqs. (15) to (22) into account, application of the requirement $\partial u/\partial m_o = 0$ to Eq. (13) yields

$$\begin{aligned} &\alpha C_1 \int_o^{\varphi_f} m_h \, d\varphi + \int_{\varphi_f}^{\varphi_o} \gamma C_{1_f} \, m_f \, d\varphi + \beta C_2 \int_{\varphi_o}^{\pi} m_H \, d\varphi \\ &- \frac{C_3}{\alpha} (t_o\alpha - m_o)\varphi_f - (t_o - m_o) \int_{\varphi_f}^{\varphi_o} \frac{C_{3_f}}{\gamma} \, d\varphi \\ &- \frac{C_4}{\beta} (t_o\alpha - m_o)(\pi - \varphi_o) = 0 \end{aligned} \tag{23}$$

and application of the requirement $\partial u/\partial t_o$ yields

$$- \alpha^2 C_1 \int_0^{\varphi_f} m_h(1 - \cos\varphi)\, d\varphi - \int_{\varphi_f}^{\varphi_o} \gamma C_{1_f}\, m_f(\alpha - \gamma\cos\varphi)\, d\varphi$$

$$- \beta C_2 \int_{\varphi_o}^{\pi} m_H(\alpha - \beta\cos\varphi)\, d\varphi + C_3(t_o\alpha - m_o)\, \varphi_f \tag{24}$$

$$+ \alpha(t_o\alpha - m_o) \int_{\varphi_f}^{\varphi_o} \frac{C_{3_f}}{\gamma}\, d\varphi + \frac{\alpha}{\beta} C_4(t_o\alpha - m_o)(\pi - \varphi_o) = 0$$

If Eq. (23) is multiplied by α and added to (24), the following simplification results:

$$\alpha^2 C_1 \int_0^{\varphi_f} m_h \cos\varphi\, d\varphi + \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f}\, m_f \cos\varphi\, d\varphi$$

$$+ \beta^2 C_2 \int_{\varphi_o}^{\pi} m_H \cos\varphi\, d\varphi = 0 \tag{25}$$

Equations (23) and (25) may be integrated by substituting the values of m_h, m_f, and m_H from Eqs. (1), (2), and (3). Of course, any integral involving quantities with the subscript f cannot be integrated until a specific function is assigned which governs the geometry of the transition section. We leave this to a later section.

The results of integrating Eqs. (23) and (25) at this point wherever possible are given in the following expressions:

$$[(\varphi_f \alpha C_1 + \delta\beta C_2) + (\varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta}) + \int_{\varphi_f}^{\varphi_o} (\gamma C_{1_f} + \frac{C_{3_f}}{\gamma})\, d\varphi] m_o$$

$$+ [- \alpha(\varphi_f \alpha C_1 + \delta\beta C_2) + (\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o) - \alpha\, (\varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta})$$

$$- \alpha \int_{\varphi_f}^{\varphi_o} (\gamma C_{1_f} + \frac{C_{3_f}}{\gamma})\, d\varphi + \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos \varphi\, d\varphi] t_o \tag{26}$$

$$= [(\alpha^2 \varphi_f C_1 + \delta\beta^2 C_2) - (\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o)$$

$$+ \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} (1 - \cos \varphi)\, d\varphi] a_o$$

$$[(\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o) + \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos \varphi d\varphi] m_o$$

$$+ [- \alpha(\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o) + \frac{1}{2} (\alpha^3 \varphi_f C_1 + \beta^3 \delta C_2)$$

$$+ \frac{1}{4} (\alpha^3 C_1 \sin 2\varphi_f - \beta^3 C_2 \sin 2\varphi_o) - \alpha \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos \varphi d\varphi$$

$$+ \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos^2 \varphi\, d\varphi] t_o = [(\alpha^3 C_1 \sin \varphi_f - \beta^3 C_2 \sin \varphi_o) \tag{27}$$

$$- \frac{1}{2} (\alpha^3 \varphi_f + \beta^3 \delta C_2)$$

$$- \frac{1}{4} (\alpha^3 C_1 \sin 2\varphi_f - \beta^3 C_2 \sin 2\varphi_o) + \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos \varphi d\varphi$$

$$- \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos^2 \varphi d\varphi] a_o$$

where

$$\delta = \pi(1 - \frac{\varphi_o}{\pi}) \tag{28}$$

The solutions of the simultaneous equations (26) and (27) are

$$m_o = \frac{1}{\Delta} \{- \delta\beta(\alpha - \beta)C_2 \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos^2\varphi d\varphi$$

$$+ \alpha^2\beta(\alpha - \beta)(\alpha\delta \sin \varphi_f + \beta\varphi_f \sin \varphi_o)C_1C_2$$

$$- \frac{\delta}{2} \beta(\alpha - \beta)(\alpha^3\varphi_f C_1 + \beta^3\delta C_2)C_2$$

$$- \frac{\delta}{4} \beta(\alpha - \beta)(\alpha^3 C_1 \sin 2\varphi_f - \beta^3 C_2 \sin 2\varphi_o)C_2$$

$$- \beta^2(\alpha - \beta)(\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o)C_2 \sin \varphi_o$$

$$+ [- \alpha(\alpha^2\varphi_f C_1 + \beta^2\delta C_2) + \alpha(\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o)$$

$$- \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos \varphi d\varphi - \alpha \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f}(1 - \cos \varphi)d\varphi$$

$$- \beta^2(\alpha - \beta)C_2 \sin \varphi_o] \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos \varphi d\varphi$$

$$+ [- \alpha(\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o) + \frac{1}{2} (\alpha^3\varphi_f C_1 + \beta^3\delta C_2)$$

$$+ \frac{1}{4} (\alpha^3 C_1 \sin 2\varphi_f - \beta^3 C_2 \sin 2\varphi_o)$$

$$+ \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos^2\varphi d\varphi] \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} d\varphi$$

$$+ [- \alpha(\varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta}) - \alpha \int_{\varphi_f}^{\varphi_o} (\gamma C_{1_f} + \frac{C_{3_f}}{\gamma}) d\varphi] \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos^2\varphi d\varphi$$

$$+ [\alpha \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos \varphi d\varphi + \alpha(\alpha^3 C_1 \sin \varphi_f - \beta^3 C_2 \sin \varphi_o)$$

$$- \frac{\alpha}{2}(\alpha^3 \varphi_f C_1 + \beta^3 \delta C_2) - \frac{\alpha}{4}(\alpha^3 C_1 \sin 2\varphi_f$$

$$- \beta^3 C_2 \sin 2\varphi_o)] \int_{\varphi_f}^{\varphi_o} (\gamma C_{1_f} + \frac{C_{3_f}}{\gamma}) d\varphi$$

$$+ \alpha(\varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta})(\alpha^3 C_1 \sin \varphi_f - \beta^3 C_2 \sin \varphi_o)$$

$$- \frac{\alpha}{2} (\varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta})(\alpha^3 \varphi_f C_1 + \beta^3 \delta C_2)$$

$$- \frac{\alpha}{4} (\varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta})(\alpha^3 C_1 \sin 2\varphi_f - \beta^3 C_2 \sin 2\varphi_o)$$

$$+ [\alpha(\varphi_f \alpha C_1 + \delta\beta C_2) - (\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o)$$

$$+ \alpha(\varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta})] \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos \varphi d\varphi\} a_o \qquad (29)$$

and

$$t_o = -\,a_o + \frac{1}{\Delta}\{\alpha\beta(\alpha\delta \sin\varphi_f + \beta\varphi_f \sin\varphi_o)(\alpha - \beta)C_1C_2$$

$$+ (\varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta})(\alpha^3C_1 \sin\varphi_f - \beta^3C_2 \sin\varphi_o)$$

$$- [-(\alpha^3C_1 \sin\varphi_f - \beta^3C_2 \sin\varphi_o)$$

$$- \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos\varphi d\varphi] \int_{\varphi_f}^{\varphi_o} (\gamma C_{1_f} + \frac{C_{3_f}}{\gamma})d\varphi$$

$$- (\alpha^2\varphi_f C_1 + \delta\beta^2 C_2) \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos\varphi d\varphi$$

$$+ (\varphi_f\alpha C_1 + \delta\beta C_2 + \varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta}) \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos\varphi\, d\varphi\} a_o \quad (30)$$

where

$$\Delta = \frac{1}{2}(\alpha^3\varphi_f C_1 + \beta^3\delta C_2)(\varphi_f\alpha C_1 + \delta\beta C_2 + \varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta})$$

$$+ \frac{1}{4}(\alpha^3C_1 \sin 2\varphi_f - \beta^3C_2 \sin 2\varphi_o)(\varphi_f\alpha C_1 + \delta\beta C_2 + \varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta})$$

$$- (\alpha^2C_1 \sin\varphi_f - \beta^2C_2 \sin\varphi_o)^2$$

$$+ [\frac{1}{2}(\alpha^3\varphi_f C_1 + \beta^3\delta C_2) + \frac{1}{4}(\alpha^3C_1 \sin 2\varphi_f - \beta^3C_2 \sin 2\varphi_o)$$

$$+ \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos^2\varphi \, d\varphi] \int_{\varphi_f}^{\varphi_o} (\gamma C_{1_f} + \frac{C_{3_f}}{\gamma}) \, d\varphi$$

$$- 2(\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o) \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos \varphi d\varphi$$

$$- [\int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos \varphi d\varphi]^2 \tag{31}$$

By combining Eqs. (29), (30), and (31) with Eqs. (1), (2), and (3) the expression for m_h, m_f, m_H as a function of a_o can be stated. The writing of these expressions is simplified if we let

$$K_1 = \{\frac{\beta}{\Delta} (\alpha - \beta) C_2 [- \delta \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos^2\varphi d\varphi$$

$$- \frac{\delta}{2} (\alpha^3 \varphi_f C_1 + \beta^3 \delta C_2) - \beta \sin \varphi_o \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos \varphi d\varphi$$

$$- \frac{\delta}{4} (\alpha^3 C_1 \sin 2\varphi_f - \beta^3 C_2 \sin 2\varphi_o)$$

$$- \beta(\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o) \sin \varphi_o]$$

$$+ \frac{1}{\Delta} [\alpha(\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o)$$

$$+ \alpha \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos \varphi d\varphi - \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos \varphi d\varphi] \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos \varphi d\varphi$$

$$+ \frac{1}{\Delta} [\frac{1}{2} (\alpha^3 \varphi_f C_1 + \beta^3 \delta C_2) + \frac{1}{4} (\alpha^3 C_1 \sin 2\varphi_f - \beta^3 C_2 \sin 2\varphi_o)$$

$$+ \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos^2\varphi d\varphi] \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} d\varphi$$

$$+ \frac{\alpha}{\Delta} [(\varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta}) + \int(\gamma C_{1_f} + \frac{C_{3_f}}{\gamma}) d\varphi] \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos^2\varphi d\varphi$$

$$- \frac{\alpha}{2\Delta} [(\alpha^3 \varphi_f C_1 + \beta^3 \delta C_2) + \frac{1}{2} (\alpha^3 C_1 \sin 2\varphi_f$$

$$- \beta^3 C_2 \sin 2\varphi_o)] \int_{\varphi_f}^{\varphi_o} (\gamma C_{1_f} + \frac{C_{3_f}}{\gamma}) d\varphi$$

$$- \frac{\alpha}{2\Delta} [(\dot{\alpha}^3 \varphi_f C_1 + \beta^3 \delta C_2) + \frac{1}{2} (\alpha^3 C_1 \sin 2\varphi_f$$

$$- \beta^3 C_2 \sin 2\varphi_o)](\varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta})$$

$$- \frac{1}{\Delta} (\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o) \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos \varphi d\varphi\} \tag{32}$$

and

$$K_2 = \frac{\alpha}{\Delta} \{\alpha\beta(\alpha - \beta)(\alpha\delta \sin \varphi_f + \beta\varphi_f \sin \varphi_o) C_1 C_2$$

$$+ (\varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta})(\alpha^3 C_1 \sin \varphi_f - \beta^3 C_2 \sin \varphi_o)$$

$$+ [(\alpha^3 C_1 \sin \varphi_f - \beta^3 C_2 \sin \varphi_o)$$

$$+ \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos \varphi d\varphi] \int_{\varphi_f}^{\varphi_o} (\gamma C_{1_f} + \frac{C_{3_f}}{\gamma}) d\varphi$$

$$- [(\alpha^2 C_1 \sin\varphi_f - \beta^2 C_2 \sin\varphi_o)$$

$$+ \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos\varphi d\varphi] \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f}\, d\varphi$$

$$- (\alpha^2 \varphi_f C_1 + \delta\beta^2 C_2) \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos\varphi d\varphi$$

$$+ (\varphi_f \alpha C_1 + \delta\beta C_2 + \varphi_f \frac{C_3}{\alpha} + \delta \frac{C_4}{\beta}) \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos\varphi d\varphi\} \tag{33}$$

so that

$$(m_h/a_o) = K_1 + K_2 \cos\varphi \qquad 0 \leq \varphi \leq \varphi_f \tag{34}$$

$$(m_f/a_o) = K_1 + (\alpha - \gamma) + (\gamma/\alpha) K_2 \cos\varphi \qquad \varphi_f \leq \varphi \leq \varphi_o \tag{35}$$

$$(m_H/a_o) = K_1 + (\alpha - \beta) + (\beta/\alpha) K_2 \cos\varphi \qquad \varphi_o \leq \varphi \leq \pi \tag{36}$$

Eqs. (32) and (33) are not yet amenable to a numerical determination. There remain within these equations terms of the type

$$\alpha \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos\varphi d\varphi - \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos\varphi d\varphi \ .$$

Because the magnitudes of α and γ are very close to unity, these two terms combine to produce a small-difference which cannot be eliminated until the geometry of the transition section is specified. Further discussion of terms of the type mentioned above is given in the Appendices.

4. GEOMETRY OF THE RING

The ring under investigation is fabricated of three distinct sections, (Fig. 1). Because of the symmetry of the ring with respect to a vertical axis, this discussion will refer only to that part of the ring between $\varphi = 0$ and $\varphi = \pi$ (see Fig. 1).

Of the three segments of the ring, two are uniform. One uniform segment lies between $\varphi = 0$ and $\varphi = \varphi_f$, and the other between $\varphi = \varphi_o$ and $\varphi = \pi$ (see Fig. 1 for definition of φ_f and φ_o). The transition section between the two uniform sections, however, must vary smoothly, i.e., the cross sectional areas are continuous at the two junctions but their derivatives are not. Furthermore, a choice of geometry for the transition section will determine whether or not the integrals in Eqs. (32) and (33) are easily integrable.

There are, in Eqs. (32) and (33), five integrals which must be evaluated. These are

$$I_1 = \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos \varphi d\varphi \tag{37}$$

$$I_2 = \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos^2 \varphi d\varphi \tag{38}$$

$$I_3 = \int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} d\varphi \tag{39}$$

$$I_4 = \int_{\varphi_f}^{\varphi_o} (\gamma C_{1_f} + \frac{C_{3_f}}{\gamma})\, d\varphi \tag{40}$$

$$I_5 = \int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos \varphi d\varphi \tag{41}$$

The quantities γ, C_{1_f}, and C_{3_f} must be stipulated as functions of φ. The choice of these functions should be based on the following criteria: 1) the resulting shape of the transition section and whether it is physically reasonable, and 2) the degree of complexity of the integrals (37) to (41). With these criteria in mind the functions

$$\gamma = \alpha(\frac{\alpha}{\beta})^{\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o}} \tag{42}$$

$$C_{1_f} = C_1(\frac{C_1}{C_2})^{\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o}} \tag{43}$$

and

$$C_{3_f} = C_3(\frac{C_3}{C_4})^{\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o}} \tag{44}$$

where chosen. It should be realized that these functions satisfy

the imposed continuity conditions, namely,

$$\begin{aligned} &\varphi = \varphi_f \\ &\qquad \gamma = \alpha \\ &\qquad C_{1_f} = C_1 \\ &\qquad C_{3_f} = C_3 \end{aligned} \tag{45}$$

$$\begin{aligned} &\varphi = \varphi_o \\ &\qquad \gamma = \beta \\ &\qquad C_{1_f} = C_2 \\ &\qquad C_{3_f} = C_4 \end{aligned} \tag{46}$$

With the functions (42) to (44) defined, the integrals I_1 to I_5 may now be evaluated. Substitution of Eqs. (42) to (44) into Eqs. (37) to (41) yields the following results:

$$I_1 = - \frac{(\varphi_f - \varphi_o)\ln \dfrac{\alpha^2 C_1}{\beta^2 C_2}}{\left(\ln \dfrac{\alpha^2 C_1}{\beta^2 C_2}\right)^2 + (\varphi_f - \varphi_o)^2} (\alpha^2 C_1 \cos \varphi_f - \beta^2 C_2 \cos \varphi_o)$$

$$- \frac{(\varphi_f - \varphi_o)^2}{(\ln \frac{\alpha^2 C_1}{\beta^2 C_2})^2 + (\varphi_f - \varphi_o)^2} (\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o) \quad (47)$$

$$I_2 = - \frac{1}{8} \frac{(\varphi_f - \varphi_o) \ln \frac{\alpha^3 C_1}{\beta^3 C_2}}{\frac{1}{4} (\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2 + (\varphi_f - \varphi_o)^2} (\alpha^3 C_1 \cos 2\varphi_f - \beta^3 C_2 \cos 2\varphi_o)$$

$$- \frac{1}{4} \frac{(\varphi_f - \varphi_o)^2}{\frac{1}{4} (\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2 + (\varphi_f - \varphi_o)^2} (\alpha^3 C_1 \sin 2\varphi_f - \beta^3 C_2 \sin 2\varphi_o)$$

$$- \frac{1}{2} \frac{(\varphi_f - \varphi_o)}{\ln \frac{\alpha^3 C_1}{\beta^3 C_2}} (\alpha^3 C_1 - \beta^3 C_2) \quad (48)$$

$$I_3 = - \frac{(\varphi_f - \varphi_o)}{\ln \frac{\alpha^2 C_1}{\beta^2 C_2}} (\alpha^2 C_1 - \beta^2 C_2) \quad (49)$$

$$I_4 = -\frac{(\varphi_f - \varphi_o)}{\ell n\,\frac{\alpha C_1}{\beta C_2}}\,(\alpha C_1 - \beta C_2) - \frac{(\varphi_f - \varphi_o)}{\ell n\,\frac{C_3/\alpha}{C_4/\beta}}\,(\frac{C_3}{\alpha} - \frac{C_4}{\beta}) \tag{50}$$

$$I_5 = -\frac{(\varphi_f - \varphi_o)\,\ell n\frac{\alpha^3 C_1}{\beta^3 C_2}}{(\ell n\,\frac{\alpha^3 C_1}{\beta^3 C_2})^2 + (\varphi_f - \varphi_o)^2}\,(\alpha^3 C_1 \cos\varphi_f - \beta^3 C_2 \cos\varphi_o)$$

$$-\frac{(\varphi_f - \varphi_o)^2}{(\ell n\,\frac{\alpha^3 C_1}{\beta^3 C_2})^2 + (\varphi_f - \varphi_o)^2}\,(\alpha^3 C_1 \sin\varphi_f - \beta^3 C_2 \sin\varphi_o) \tag{51}$$

5. BENDING STRESSES

In order to obtain the bending stresses in the ring the following procedure should be pursued:

(a) Obtain a_o by solving for the interaction load between an infinitely long hydrostatically loaded cylinder and a single uniform reinforcing ring (or any other approximation of Ref. [4]),

$$a_o = \frac{2.740(a/h)^{-1/2}}{1.610\ (a/h)^{-3/2}\ \dfrac{C_1^o\ C_3^o}{C_1^o + C_3^o} + 1}\ \frac{p}{2E} \tag{52}$$

where the uniform ring is selected such that

$$\begin{aligned} C_1^o &= \frac{1}{2}\ (C_1 + C_2) \\ C_3^o &= \frac{1}{2}\ (C_3 + C_4) \end{aligned} \tag{53}$$

(b) Evaluate integrals I_1 to I_5 from Eqs. (47) to (51).

(c) Evaluate the quantity $I_1[\alpha_1 I_1 - I_5]$ from Eqs. (A8), (A11) and (A12) (See Appendix). Evaluate the quantity $I_5 \int_{\varphi_f}^{\varphi_o} \gamma C_{1_f}\, d\varphi - I_1 I_3$ from Eqs. (A8), (A11), (A13), (A14), (A15). Evaluate the quantity $\alpha - \gamma$ from Eqs. (B10) and (B11).

(d) Utilizing the results of (a), (b) and (c), above, evaluate Eqs. (31) to (36).

(e) Obtain the bending stress distribution in the ring at the radius a (the outer radius) from the relations

$$\left.\frac{(\sigma_\varphi)_b}{p}\right|_{\text{Outer Radius}} = \frac{(1-\alpha)}{\alpha} C_1 \, m_h \qquad 0 \le \varphi \le \varphi_f$$

$$\left.\frac{(\sigma_\varphi)_b}{p}\right|_{\text{Outer Radius}} = \frac{(1-\gamma)}{\gamma} C_{1_f} \, m_f \qquad \varphi_f \le \varphi \le \varphi_o \tag{54}$$

and

$$\left.\frac{(\sigma_\varphi)_b}{p}\right|_{\text{Outer Radius}} = \frac{(1-\beta)}{\beta} C_2 \, m_H \qquad \varphi_o \le \varphi \le \pi$$

6. RESULTS

In order to judge the effect of the transition section in reducing the maximum bending stress in the ring, a single set of parameters describing the geometry of the ring was chosen. However, many different lengths of the transition section were considered.

The particular set of parameters which determine the geometry of the ring (an inner ring) are

$$C_1 = 9.983(10^7) \qquad C_2 = 5.467(10^7)$$

$$C_3 = 2.750(10^3) \qquad C_4 = 2.250(10^3)$$

$$\alpha = 0.9909091 \qquad \beta = 0.9888889$$

and those which determine the length of the transition section of the rings are

$$\varphi_o = 90^\circ$$

$$\varphi_f = 90^\circ,\ 80^\circ,\ 70^\circ,\ 60^\circ,\ 45^\circ,\ 30^\circ,\ 10^\circ,\ 0^\circ.$$

The results obtained from the procedure outlined in

section 5 are presented in the form of curves in Figs. 5 and 6 . The maximum bending stress distributions are plotted in Fig. 5 for different lengths of transition section.

In addition to the stress distribution curves, two additional curves are presented in Fig. 6. They represent the reduction in the peak stresses, non-dimensionalized with respect to the corresponding peak stress with no transition section, versus the length of transition section. The solid curve represents the stress reduction at the foot of the transition section ($\varphi = \varphi_f$). The other curve represents the stress reduction at the head of the transition section ($\varphi = \varphi_o$).

CONCLUSIONS

The primary purpose of the present analysis has been to determine the reduction in the maximum bending stress caused by introducing a smooth transition between two uniform but different sections of a shell reinforcing ring. The non-uniform ring without the discontinuity has previously been analyzed [1]. In these previous analyses the bending stress at the discontinuity was found to be of the order of one third the membrane stress because of the eccentricity at the discontinuity of the centroidal radius of the ring. In Fig. 5 the bending stresses in the ring without any transition section correspond to the $\varphi_o - \varphi_f = 0^o$ curve.

The effect of the length of the transition section $(\varphi_o - \varphi_f)$ is indicated in both Fig. 5, where the stress distributions are plotted for the various transition section lengths, and in Fig. 6, where the stress reduction is plotted vs. the length of transition section. These curves show a reduction in the maximum bending stress of approximately 60% in the less rigid section and 50% in the more rigid section of the ring. For the two uniform ring sections the greatestreduction in the maximum bending stress, however, occurs at different values of $\varphi_o - \varphi_f$. Thus, the maximum reduction in the bending stress occurs in the less rigid section when $\varphi_o - \varphi_f \approx 60^o$; the corresponding maximum stress reduction in the more rigid section occurs when $\varphi_o - \varphi_f = 90^o$.

The reduction in the bending stress reaches a maximum value in the less rigid section because the section is becoming shorter and effectively stiffer. This effect becomes significant after $\varphi_o - \varphi_f \approx 60^\circ$. For $\varphi_o - \varphi_f > 60^\circ$, the bending stress at the foot of the transition section is greater than its corresponding value for $\varphi_o - \varphi_f \approx 60^\circ$.

The results indicate that a substantial stress reduction can be obtained by introducing a transition section between the two uniform sections of the ring. They further indicate that there is an optimum length of transition section which is suggested by a maximum reduction in the bending stress.

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APPENDIX

Equations (32) and (33) contain expressions of the type

$$I_1[\alpha I_1 - I_5] = I_1\alpha\int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos\varphi d\varphi - I_1\int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos\varphi d\varphi \qquad (A1)$$

and

$$I_5\int_{\varphi_f}^{\varphi_o} \gamma C_{1_f} d\varphi - I_1 I_3 = [\int_{\varphi_f}^{\varphi_o} \gamma^3 C_{1_f} \cos\varphi d\varphi][\int_{\varphi_f}^{\varphi_o} \gamma C_{1_f} d\varphi]$$

$$- [\int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} \cos\varphi d\varphi][\int_{\varphi_f}^{\varphi_o} \gamma^2 C_{1_f} d\varphi] \qquad (A2)$$

Because the magnitudes of α and γ are close to unity, Eqs. (A1) and (A2) represent differences of nearly equal quantities. These differences, which could lead to the loss of significant figures in a numerical analysis, are eliminated by appropriate expansions.

If use is made of Eqs. (37) and (41), Eq. (A1) becomes

$$\alpha I_1{}^2 - I_1 I_5 =$$

$$I_1\{-(\varphi_f - \varphi_o)[\frac{\alpha \ln \frac{\alpha^2 C_1}{\beta^2 C_2}}{(\ln \frac{\alpha^2 C_1}{\beta^2 C_2})^2 + (\varphi_f - \varphi_o)^2} (\alpha^2 C_1 \cos\varphi_f - \beta^2 C_2 \cos\varphi_o)$$

$$- \frac{\ln \frac{\alpha^3 C_1}{\beta^3 C_2}}{(\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2 + (\varphi_f - \varphi_o)^2} (\alpha^3 C_1 \cos \varphi_f - \beta^3 C_2 \cos \varphi_o)]$$

$$-(\varphi_f - \varphi_o)^2 [\frac{\alpha}{(\ln \frac{\alpha^2 C_1}{\beta^2 C_2})^2 + (\varphi_f - \varphi_o)^2} (\alpha^2 C_1 \sin \varphi_f - \beta^2 C_2 \sin \varphi_o)$$

$$- \frac{1}{(\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2 + (\varphi_f - \varphi_o)^2} (\alpha^3 C_1 \sin \varphi_f - \beta^3 C_2 \sin \varphi_o)]\} \qquad \text{(A1a)}$$

It is evident that the expressions which will produce the small differences are those within the square brackets. Consider then the term

$$\frac{\alpha \ln \frac{\alpha^2 C_1}{\beta^2 C_2}}{(\ln \frac{\alpha^2 C_1}{\beta^2 C_2})^2 + (\varphi_f - \varphi_o)^2} (\alpha^2 C_1 \cos \varphi_f - \beta^2 C_2 \cos \varphi_o)$$

$$- \frac{\ln \frac{\alpha^3 C_1}{\beta^3 C_2}}{(\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2 + (\varphi_f - \varphi_o)^2} (\alpha^3 C_1 \cos \varphi_f - \beta^3 C_2 \cos \varphi_o) \qquad \text{(A3)}$$

Expression (A3) may be rewritten

$$\frac{\ln \frac{\alpha^3 C_1}{\beta^3 C_2}}{(\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2 + (\varphi_f - \varphi_o)^2} \{ [\frac{(\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2 + (\varphi_f - \varphi_o)^2}{(\ln \frac{\alpha^2 C_1}{\beta^2 C_2})^2 + (\varphi_f - \varphi_o)^2}] \frac{\ln \frac{\alpha^2 C_1}{\beta^2 C_2}}{\ln \frac{\alpha^3 C_1}{\beta^3 C_2}} (\alpha^2 C_1 \cos \varphi_f$$

$$- \beta^2 C_2 \cos \varphi_o)\alpha - (\alpha^3 C_1 \cos \varphi_f - \beta^3 C_2 \cos \varphi_o) \} \qquad \text{(A3a)}$$

Let

$$r_1 = \frac{\ln \frac{\alpha^2 C_1}{\beta^2 C_2}}{\ln \frac{\alpha^3 C_1}{\beta^3 C_2}} \qquad \text{(A4)}$$

Now expression (A3a) becomes upon substitution of (A4)

$$\frac{\ln \frac{\alpha^3 C_1}{\beta^3 C_2}}{(\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2 + (\varphi_f - \varphi_o)^2} \left\{ \frac{1 + \frac{(\varphi_f - \varphi_o)^2}{(\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2}}{r_1^2 + \frac{(\varphi_f - \varphi_o)^2}{(\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2}} r_1 \; (\alpha^3 C_1 \cos \varphi_f \right.$$

$$\left. - \alpha\beta^2 C_2 \cos \varphi_o) - (\alpha^3 C_1 \cos \varphi_f - \beta^3 C_2 \cos \varphi_o) \right\} \tag{A5}$$

It can be seen that if α and β are quantities whose value are close to unity, r_1 is a quantity of order of magnitude unity, and can be written

$$r_1 = 1 + \varepsilon_1 \tag{A6}$$

where ε_1 is small compared to unity. Expression (A5) can again be rewritten as

$$\frac{\ln \frac{\alpha^3 C_1}{\beta^3 C_2}}{(\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2 + (\varphi_f - \varphi_o)^2} \left\{ \frac{1 + \varepsilon_1}{1 + \frac{\varepsilon_1 (2 + \varepsilon_1)}{1 + \frac{(\varphi_f - \varphi_o)^2}{(\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2}}} (\alpha^3 C_1 \cos \varphi \right.$$

$$- \alpha\beta^2 C_2 \cos \varphi_o) - (\alpha^3 C_1 \cos \varphi_f - \beta^3 C_2 \cos \varphi_o)\} \tag{A7}$$

Let

$$G = 1 + \frac{(\varphi_f - \varphi_o)^2}{(\ell n \dfrac{\alpha^3 C_1}{\beta^3 C_2})^2} \tag{A9}$$

and from expression (A7) let us expand

$$\frac{1 + \varepsilon_1}{1 + \dfrac{\varepsilon_1 (2 + \varepsilon_1)}{G}} = (1 + \varepsilon_1)[1 - \frac{\varepsilon_1 (2 + \varepsilon_1)}{G} + \frac{\varepsilon_1^{\,2} (2 + \varepsilon_1)^2}{G^2}$$

$$- \frac{\varepsilon_1^{\,3} (2 + \varepsilon_1)^3}{G^3} + \cdots\cdot] \tag{A9}$$

After multiplying,(A9) becomes

$$\frac{1 + \varepsilon_1}{1 + \dfrac{\varepsilon_1 (2 + \varepsilon_1)}{G}} = 1 + \varepsilon_1 (1 - \frac{2 + \varepsilon_1}{G})[1 - \frac{\varepsilon_1 (2+\varepsilon_1)}{G} + \frac{\varepsilon_1^{\,2} (2+\varepsilon_1)^2}{G^2} - \ldots] \tag{A9a}$$

In order to evaluate ε_1 let us write Eq. (A6) and (A4) in the following form and expand:

$$r_1 = (1 + \varepsilon_1) = \frac{\ell n \dfrac{\alpha^2 C_1}{\beta^2 C_2}}{\ell n \dfrac{\alpha^2 C_1}{\beta^2 C_2} + \ell n \dfrac{\alpha}{\beta}} = \frac{1}{1 + \dfrac{\ell n \frac{\alpha}{\beta}}{\ell n \frac{\alpha^2 C_1}{\beta^2 C_2}}}$$

$$= 1 - \frac{\ell n \frac{\alpha}{\beta}}{\ell n \frac{\alpha^2 C_1}{\beta^2 C_2}} + \left(\frac{\ell n \frac{\alpha}{\beta}}{\ell n \frac{\alpha^2 C_1}{\beta^2 C_2}}\right)^2 - \left(\frac{\ell n \frac{\alpha}{\beta}}{\ell n \frac{\alpha^2 C_1}{\beta^2 C_2}}\right)^3 + \cdots \tag{A10}$$

$$\varepsilon_1 = - \frac{\ell n \frac{\alpha}{\beta}}{\ell n \frac{\alpha^2 C_1}{\beta^2 C_2}} \left[1 - \frac{\ell n \frac{\alpha}{\beta}}{\ell n \frac{\alpha^2 C_1}{\beta^2 C_2}} + \left(\frac{\ell n \frac{\alpha}{\beta}}{\ell n \frac{\alpha^2 C_1}{\beta^2 C_2}}\right)^2 - \cdots\right] \tag{A11}$$

If we now substitute (A9) into (A7) and (A7) into (A1a), the following expression results:

$$I_1[\alpha I_1 - I_5] = \frac{- I_1(\varphi_f - \varphi_o)}{G \ \ell n \dfrac{\sigma^3 C_1}{\beta^3 C_2}} \{\alpha \varepsilon_1 (1 - \frac{2 + \varepsilon_1}{G})[1$$

$$- \frac{\varepsilon_1(2 + \varepsilon_1)}{G} + \frac{\varepsilon_1^{\,2}(2 + \varepsilon_1)^2}{G^2} - \cdots](\alpha^2 C_1 \cos \varphi_f$$

$$- \beta^2 C_2 \cos \varphi_o) - \beta^2(\alpha - \beta)C_2 \cos \varphi_o\}$$

$$- \frac{I_1(\varphi_f - \varphi_o)^2}{G(\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2} \{- \frac{\alpha\varepsilon_1(2 + \varepsilon_1)}{G} [1 - \frac{\varepsilon_1(2 + \varepsilon_1)}{G}$$

$$+ \frac{\varepsilon_1{}^2(2 + \varepsilon_1)^2}{G^2} - \cdots] - \beta^2(\alpha - \beta)C_2 \sin \varphi_o\} \quad \text{(A12)}$$

where ε_1 is given by (A11).

The expression (A2) may be treated in a similar manner to obtain

$$I_5\int_{\varphi_f}^{\varphi_o} \gamma C_{1_f}\, d\varphi - I_1 I_3 = - \frac{(\varphi_f - \varphi_o)^2}{G \ln \frac{\alpha^3 C_1}{\beta^3 C_2} \ln \frac{\alpha C_1}{\beta C_2}} \{[\varepsilon_2 - \frac{\varepsilon_1(2 + \varepsilon_1)}{G}][$$

$$1 - \frac{\varepsilon_1(2 + \varepsilon_1)}{G} + \frac{\varepsilon_1{}^2(2 + \varepsilon_1)^2}{G^2} - \cdots](\alpha^2 C_1 \cos \varphi_f - \beta^2 C_2 \cos \varphi_o)($$

$$\alpha^2 C_1 - \beta^2 C_2) + \alpha\beta(\alpha - \beta)(\alpha \cos \varphi_f - \beta \cos \varphi_o)C_1 C_2\}$$

$$\frac{-(\varphi_f - \varphi_o)^3}{G (\ln \frac{\alpha^3 C_1}{\beta^3 C_2})^2 \ln \frac{\alpha C_1}{\beta C_2}} \{[\varepsilon_3 - \frac{\varepsilon_1(2 + \varepsilon_1)}{G}][1 - \frac{\varepsilon_1(2 + \varepsilon_1)}{G}$$

$$+ \frac{\varepsilon_1{}^2(2+\varepsilon_1)^2}{G^2} - \cdots](\alpha^2C_1 \sin\varphi_f - \beta^2C_2 \sin\varphi_o)(\alpha^2C_1 - \beta^2C_2)$$

$$+ \alpha\beta(\alpha-\beta)(\alpha\sin\varphi_f - \beta\sin\varphi_o)C_1C_2\} \tag{A13}$$

where ε_1 is given by (A11) and

$$\varepsilon_2 = \frac{-2\ln\frac{\alpha}{\beta}}{\ln\frac{\alpha C_1}{\beta C_2}}[1 - (\frac{2\ln\frac{\alpha}{\beta}}{\ln\frac{\alpha C_1}{\beta C_2}}) + (\frac{2\ln\frac{\alpha}{\beta}}{\ln\frac{\alpha C_1}{\beta C_2}})^2 - \cdots] \tag{A14}$$

$$\varepsilon_3 = -(\frac{\ln\frac{\alpha}{\beta}}{\ln\frac{\alpha C_1}{\beta C_2}})[1 - (\frac{\ln\frac{\alpha}{\beta}}{\ln\frac{\alpha C_1}{\beta C_2}}) + (\frac{\ln\frac{\alpha}{\beta}}{\ln\frac{\alpha C_1}{\beta C_2}})^2 - \cdots] \tag{A15}$$

APPENDIX B

Equation (35) contains the term $(\alpha - \gamma)$. If α and γ are quantities of the order of magnitude of unity, the calculation of this term involves a difference of nearly equal quantities. The resulting loss of significant figures can be eliminated by suitable expansions. If use is made of Eq. (42), the quantity $(\alpha - \gamma)$ can be written as follows:

$$\alpha - \gamma = \alpha\left[1 - \left(\frac{\alpha}{\beta}\right)^{\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o}}\right] \qquad \text{(B1)}$$

Since α and β are of order unity, we may write

$$\left(\frac{\alpha}{\beta}\right)^{\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o}} = (1 + \varepsilon_4)^{\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o}} \qquad \text{(B2)}$$

where

$$\varepsilon << 1 \qquad \text{(B3)}$$

If the binomial expansion is used to express the right-hand side of Eq. (B2) in powers of ε_4, the following results:

$$(\frac{\alpha}{\beta})^{\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o}} = 1 + \frac{\varphi - \varphi_f}{\varphi_f - \varphi_o}\,\varepsilon_4 + \frac{\varphi - \varphi_f}{\varphi_f - \varphi_o}\,[\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o} - 1]\,\frac{1}{2!}\,\varepsilon_4^2$$

$$+ \frac{\varphi - \varphi_f}{\varphi_f - \varphi_o}\,[\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o} - 1][\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o} - 2]\,\frac{1}{3!}\,\varepsilon_4^3 + \cdots \quad (B4)$$

In order to find an expression for ε_4 let us write [see Eq. (7)]

$$(\frac{\alpha}{\beta}) = \frac{a_h}{a_H} \quad (B5)$$

This can be restated in terms of the thicknesses h and H of the two uniform sectors of the ring. Thus,

$$\frac{\alpha}{\beta} = \frac{a - (h/2)}{a - (H/2)} = \frac{2(a/H)}{2(a/h)}\,\frac{2(a/h) - 1}{2(a/H) - 1} \quad (B6)$$

Further algebraic manipulation leads to

$$\frac{\alpha}{\beta} = \frac{1 - (h/2a)}{1 - (H/2a)} \quad (B7)$$

If the denominator is expanded, we obtain

$$(1 - \frac{H}{2a})^{-1} = 1 + (\frac{H}{2a}) + (\frac{H}{2a})^2 + (\frac{H}{2a})^3 + \cdots \quad (B8)$$

Combining Eqs. (B7) and (B8) gives

$$\frac{\alpha}{\beta} = 1 + \frac{1}{2}\left(\frac{H}{a} - \frac{h}{a}\right) \sum_{k=1}^{\infty} \left(\frac{H}{a}\right)^{k-1} \frac{1}{2^{k-1}} \tag{B9}$$

However, if Eq. (B9) is compared with Eq. (B2), we obtain the following expression:

$$\varepsilon_4 = \frac{1}{2}\left(\frac{H}{a} - \frac{h}{a}\right) \sum_{k=1}^{\infty} \left(\frac{H}{a}\right)^{k-1} \frac{1}{2^{k-1}} \tag{B10}$$

Finally, from Eqs. (B1) and (B4) we can write the expression for $(\alpha - \gamma)$ in the following form:

$$\alpha - \gamma = -\alpha \left(\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o}\right) \varepsilon_4 \Big[1 + \left(\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o} - 1\right) \frac{\varepsilon_4}{2}$$

$$+ \left(\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o} - 1\right)\left(\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o} - 2\right) \frac{\varepsilon_4^2}{6}$$

$$+ \left(\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o} - 1\right)\left(\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o} - 2\right)\left(\frac{\varphi - \varphi_f}{\varphi_f - \varphi_o} - 3\right) \frac{\varepsilon_4^3}{24} + \cdots\Big] \tag{B11}$$

Therefore, from Eqs. (B10) and (B11) we can calculate $(\alpha - \gamma)$ without the loss of significant figures due to differences of nearly equal quantities.

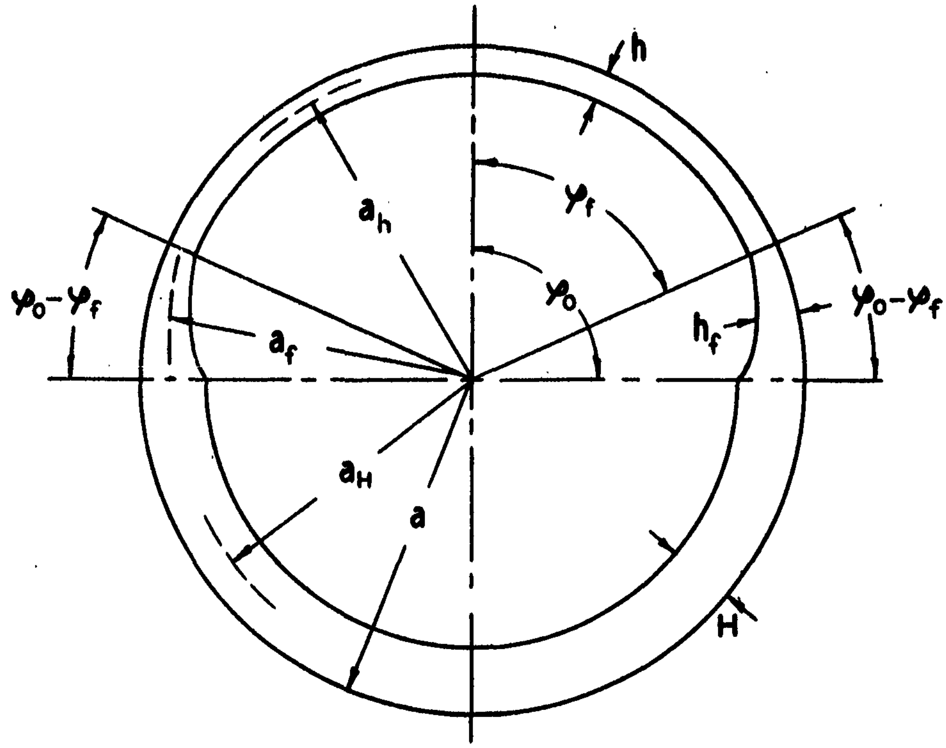

FIG. 1 RING WITH TRANSITION SECTION

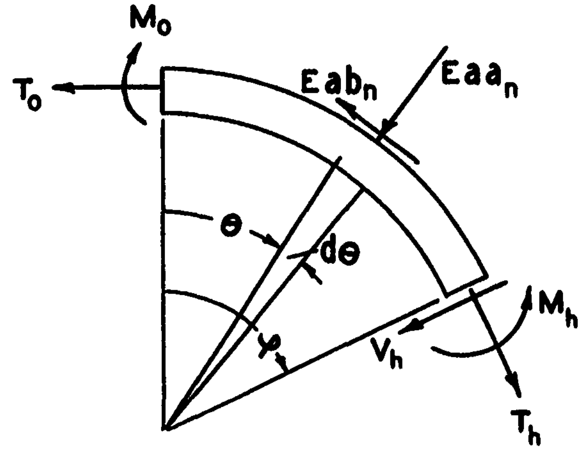

FIG. 2
FREE BODY DIAGRAM OF A SEGMENT OF THE RING, $0 \leq \varphi \leq \varphi_f$

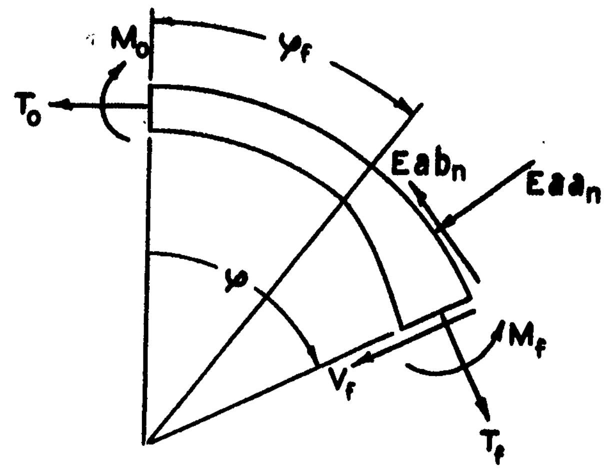

FIG. 3
FREE BODY DIAGRAM OF A SEGMENT OF THE RING, $\varphi_f \leq \varphi \leq \varphi_0$

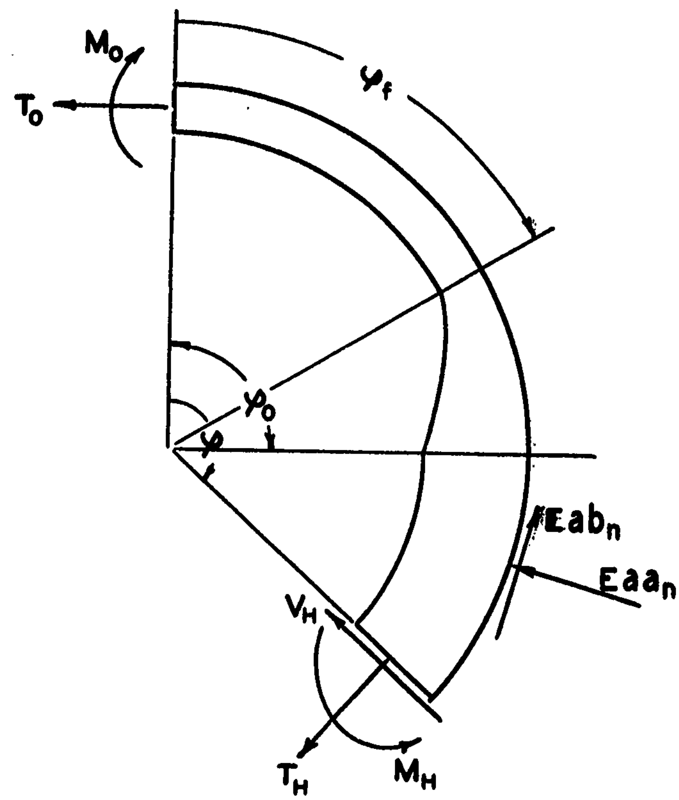

FIG. 4
FREE BODY DIAGRAM OF A SEGMENT OF THE RING, $\varphi_0 \leq \varphi \leq \pi$

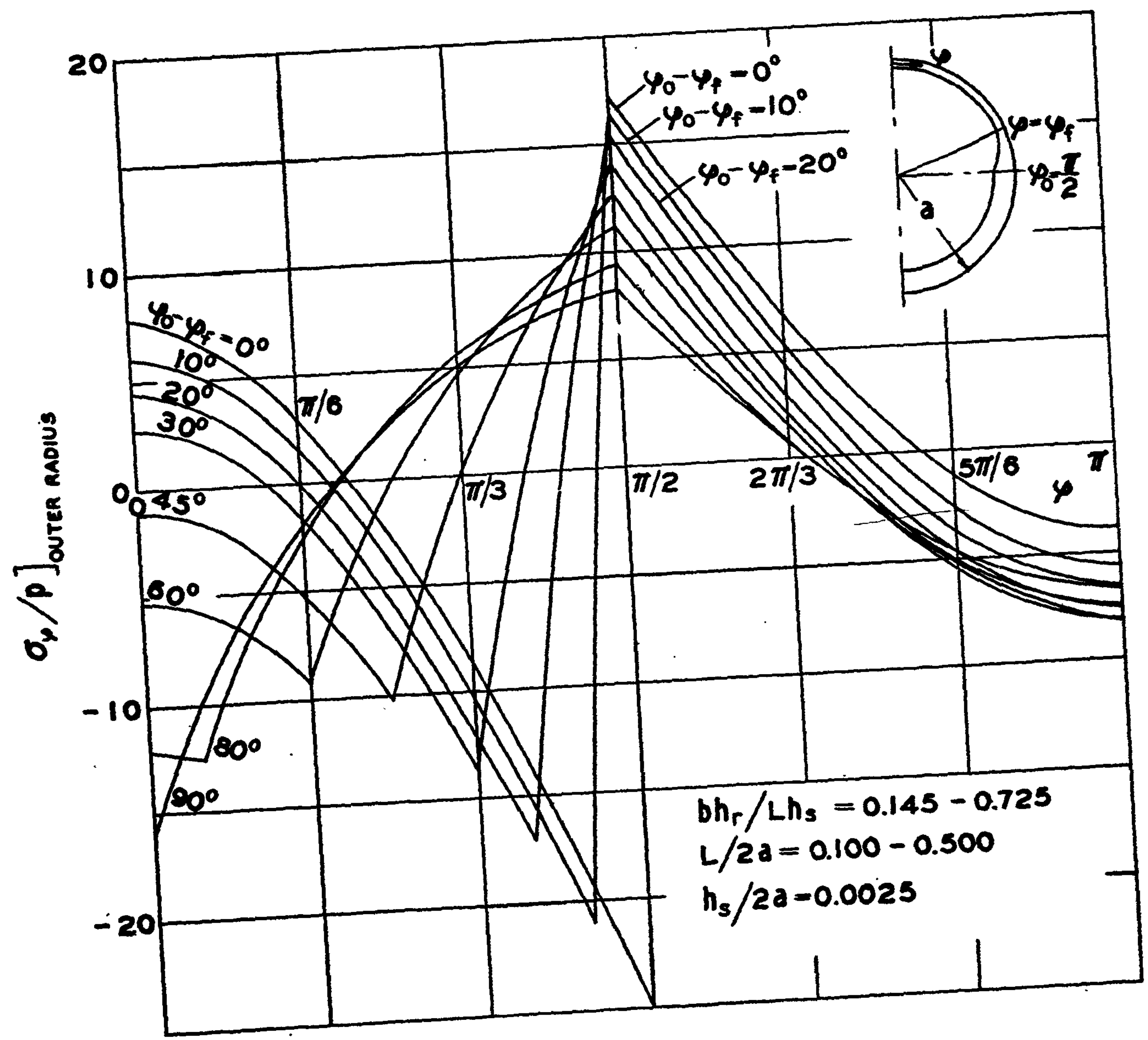

FIG. 5 BENDING STRESS DISTRIBUTION IN RING WITH TRANSITION SECTION

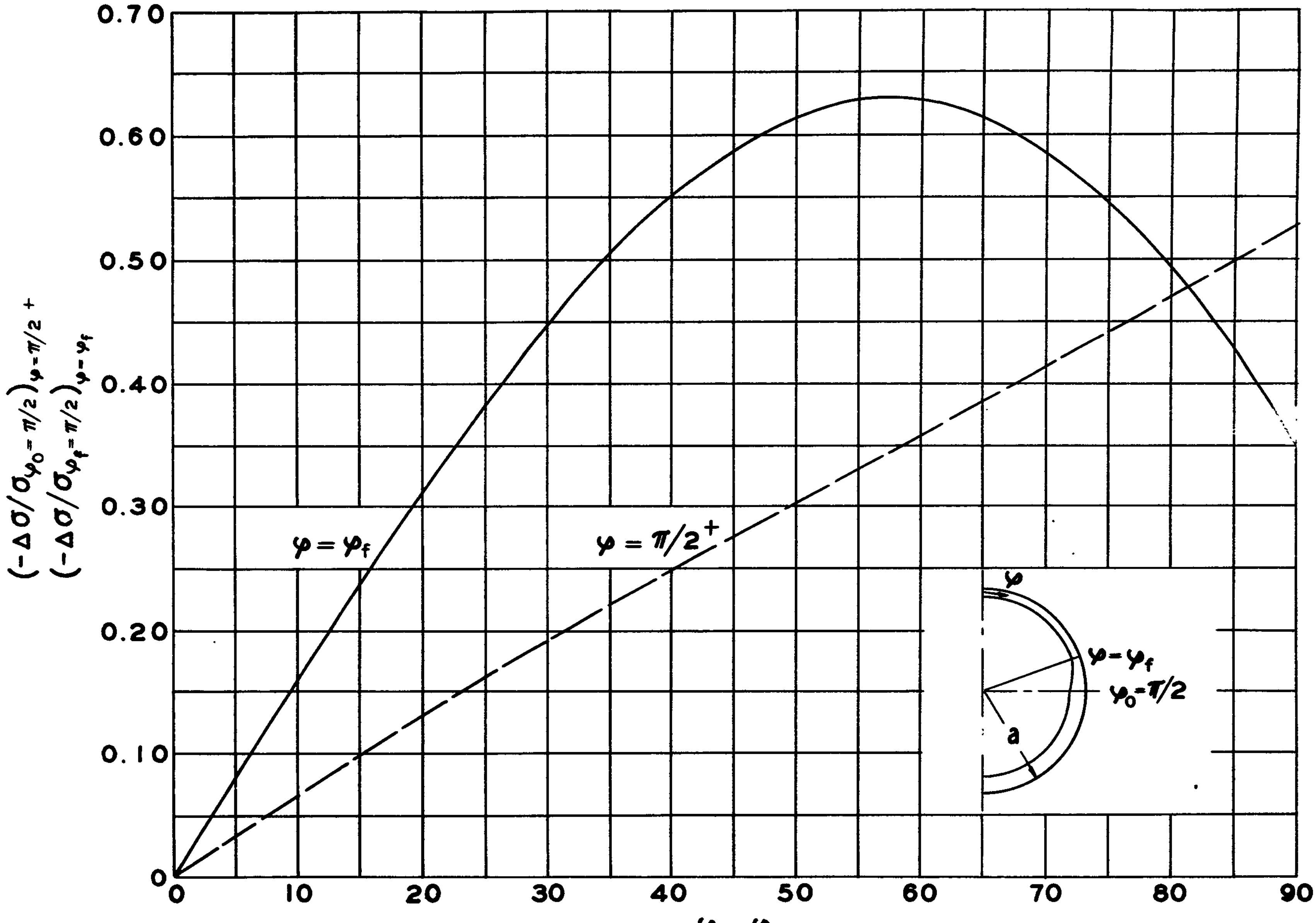

FIG. 6 STRESS REDUCTION AT $\varphi = \varphi_f$ AND $\varphi = \varphi_0$ VS LENGTH OF TRANSITION SECTION

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